

# THE DECOMPOSITION OF A LIE GROUP WITH A LEFT INVARIANT PSEUDO-RIEMANNIAN METRIC AND THE UNIQUENESS

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**ABSTRACT.** In this paper, we discuss the decomposition of a Lie group with a left invariant pseudo-Riemannian metric and the uniqueness. In fact, it is a decomposition of a Lie group into totally geodesic sub-manifolds which is different from the De Rham decomposition on a Lie group. As an application, we give a decomposition of a Lie group with a left invariant pseudo-Riemannian Einstein metric, and prove that the decomposition is unique up to the order of the parts in the decomposition.

## 0. INTRODUCTION

Lie groups play an enormous role in modern geometry. Whenever a Lie group acts on a geometric object, such as a Riemannian or a symplectic manifold, this action provides a measure of rigidity and yields a rich algebraic structure. For example, de Rham decomposition theorem exactly describes the decomposition of a pseudo-Riemannian manifold into holonomy-invariant sub-manifolds. There are a lot of studies on de Rham decompositions and holonomy groups of pseudo-Riemannian manifolds [4, 9, 13, 14, 23, 27, 28].

Let  $G$  be a Lie group with a left invariant pseudo-Riemannian metric  $\langle, \rangle$ . On one hand, we have the corresponding de Rham decomposition of  $G$ . On the other hand, since the manifold itself is a Lie group, it is also natural to discuss the decomposition of  $G$  under the action of  $G$ . The latter is closer to the structure of the Lie group. In addition, if  $\langle, \rangle$  is bi-invariant, then it is shown in [29] that the decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  into non-degenerate and irreducible Lie ideals is unique up to an isometry. But the discussion on left invariant metrics are completely different from that on bi-invariant metrics. An important reason is: the orthogonal complement of a non-degenerate Lie ideal is also a Lie ideal for a bi-invariant metric; but it doesn't hold for a left invariant metric. In [11], there are some discussions on the algebras with left invariant pseudo-Riemannian bilinear forms.

This paper is to discuss a Lie group  $G$  with a left invariant pseudo-Riemannian metric  $\langle, \rangle$ . In this paper, we discuss the local decomposition of  $G$ , i.e. the decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  into indecomposable, non-degenerate and strong ideals in terms of Levi-Cevita connections, and the uniqueness of the decomposition. Then we get the global decomposition of  $G$  or some cover group of  $G$ . Furthermore, we find that it is a decomposition of  $G$  into totally geodesic sub-manifolds. In particular, we prove the decomposition is unique up to the order if the Ricci tensor associated with  $\langle, \rangle$  is non-degenerate. It follows that the uniqueness holds for a pseudo-Riemannian Einstein metric with a non-zero constant  $c$ . There are decomposition results in a different direct for a solvable Lie group with a Riemann Einstein metric in [18], a nilpotent Lie group in [17, 25, 26] and some partial results in the compact setting in [8]. An Einstein metric is a distinguished one in the study of Riemann geometry. The classical references for Einstein manifolds are the book [7] and some expository articles [2, 5, 6, 22]. For the study of homogeneous Einstein manifolds to see [12, 15, 16, 20, 21] and so on.

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The paper is organized as follows. In Section 1, we give some definitions and facts in terms of Levi-Civita connections.

In Section 2, we give the definition of the term “indecomposable”, and then get the local decomposition of  $G$ , i.e. the decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$  into indecomposable, non-degenerate and strong ideals. Furthermore, we get the global decomposition of  $G$  corresponding to the local decomposition. In this section, we note that “decomposable” isn’t equivalent with the existence of a non-trivial, non-degenerate and strong ideal of  $\mathfrak{g}$ , which is different from that for a bi-variant metric. Also we note that an indecomposable Lie group can have a decomposition into minimal sub-manifolds.

Section 3 is to discuss the uniqueness of the decomposition. It is enough to discuss the uniqueness of the local decomposition by the discussion in Section 2.

In subsection 3.1, we show that a Lie group  $G$  such that  $Ann_R(\mathfrak{g})$  isn’t isotropic is induced from some Lie group such that  $Ann_R(\mathfrak{g})$  is isotropic.

In subsection 3.2, we prove that the local decomposition of  $G$  into indecomposable, non-degenerate and strong ideals is unique up to a strong automorphism satisfying certain conditions if  $Ann_R(\mathfrak{g})$  is isotropic.

Subsection 3.3 is to discuss the case when  $Ann_R(\mathfrak{g}) = Ann(\mathfrak{g})$ . Firstly, we give an example to show that “decomposable” doesn’t imply “orthogonal decomposable”. Then we prove that “decomposable” is equivalent with “orthogonal decomposable” if  $Ann_R(\mathfrak{g}) = Ann(\mathfrak{g})$ . Therefore there exists an orthogonal decomposition of  $\mathfrak{g}$  into indecomposable, non-degenerate and strong ideals when  $Ann_R(\mathfrak{g}) = Ann(\mathfrak{g})$ . Furthermore we prove that the orthogonal decomposition is unique up to a strong isometry.

In subsection 3.4, we discuss the Lie group satisfying  $Ann_R(\mathfrak{g}) = 0$ . It is clear that  $Ann_R(\mathfrak{g}) = Ann(\mathfrak{g}) = 0$  if  $Ann_R(\mathfrak{g}) = 0$ . Moreover if  $Ann_R(\mathfrak{g}) = 0$ , we prove that any decomposition of  $\mathfrak{g}$  is orthogonal if  $\mathfrak{g}$  is decomposable. By further discussion on the proof of the theorem in subsection 3.3, we get that the decomposition is unique up to the order of strong ideals. As a remark, we give a direct proof of this result.

In Section 4, we show that  $Ann_R(\mathfrak{g}) = 0$  if the Ricci tensor is non-degenerate. Then we get the uniqueness of the local decomposition of  $G$  following from the decomposition result in subsection 3.4. As a consequence, we get a unique decomposition result for a Lie group with a left invariant pseudo-Riemannian Einstein metric which isn’t Ricci flat. It is well known that flat is equivalent with Ricci flat for a left invariant Riemann metric. In this section, we construct some examples to show that it is false for a pseudo-Riemannian metric. At the end of this section, we note that the uniqueness doesn’t hold for the metric which is Ricci flat.

## 1. PRELIMINARY

Let  $G$  be a Lie group, let  $\mathfrak{g}$  be the Lie algebra of  $G$  consisting of all the left invariant vector fields on  $G$ , and let  $\langle, \rangle$  be a left invariant pseudo-Riemannian metric on  $G$ . Then the unique torsion-free affine connection, i.e. Levi-Civita connection, is determined by

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2} (\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle) \text{ for any } X, Y, Z \in \mathfrak{g}. \quad (1.1)$$

It is easy to see that equation (1.1) is equivalent with

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad (1.2)$$

$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle = 0 \text{ for any } X, Y, Z \in \mathfrak{g}. \quad (1.3)$$

Let  $\mathfrak{h}$  be a subspace of  $\mathfrak{g}$ . If  $\langle X, Y \rangle = 0$  for any  $X, Y \in \mathfrak{h}$ , then  $\mathfrak{h}$  is called **isotropic**. If the restriction of  $\langle, \rangle$  on  $\mathfrak{h}$  is non-degenerate, then  $\mathfrak{h}$  is called **non-degenerate**. For any subspace

$\mathfrak{h}_1$  and  $\mathfrak{h}_2$  of  $\mathfrak{g}$ , let  $\nabla_{\mathfrak{h}_1}\mathfrak{h}_2$  denote the subspace extended by  $\nabla_X Y$  for any  $X \in \mathfrak{h}_1$  and  $Y \in \mathfrak{h}_2$ . If  $\nabla_{\mathfrak{g}}\mathfrak{h} \subseteq \mathfrak{h}$  and  $\nabla_{\mathfrak{h}}\mathfrak{g} \subseteq \mathfrak{h}$ , then  $\mathfrak{h}$  is called a **strong ideal** of  $\mathfrak{g}$ . It is clear that a strong ideal is a Lie ideal.

Let  $G$  and  $G'$  be Lie groups, let  $\mathfrak{g}$  and  $\mathfrak{g}'$  be the Lie algebra of  $G$  and  $G'$ , let  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle'$  be left invariant pseudo-Riemannian metrics on  $G$  and  $G'$ , and let  $\nabla$  and  $\nabla'$  be the Levi-Civita connections respectively. A linear map  $\pi$  from  $\mathfrak{g}$  to  $\mathfrak{g}'$  is called a **strong homomorphism** if  $\pi(\nabla_X Y) = \nabla'_{\pi(X)} \pi(Y)$  for any  $X, Y \in \mathfrak{g}$ ; a **strong isomorphism** if  $\pi$  is a linear isomorphism and a strong homomorphism; a **strong automorphism** if  $\pi$  is a strong isomorphism and  $\mathfrak{g} = \mathfrak{g}'$ . An analytic homomorphism  $\Pi$  from  $G$  to  $G'$  is called a **strong homomorphism** (or **strong isomorphism**) if  $d\Pi$  is a strong homomorphism (or strong isomorphism) from  $\mathfrak{g}$  to  $\mathfrak{g}'$ . A **strong automorphism** is a strong isomorphism from  $G$  to  $G$ . A strong isomorphism  $\pi$  from  $\mathfrak{g}$  to  $\mathfrak{g}'$  is called a **strong isometry** if

$$\langle \pi(X), \pi(Y) \rangle' = \langle X, Y \rangle \text{ for any } X, Y \in \mathfrak{g};$$

A strong isomorphism  $\Pi$  from  $G$  to  $G'$  is called a **strong isometry** if  $d\Pi$  is a strong isometry from  $\mathfrak{g}$  to  $\mathfrak{g}'$ .

Let  $\mathfrak{h}^\perp$  denote the subspace of  $\mathfrak{g}$  orthogonal to  $\mathfrak{h}$  with respect to  $\langle \cdot, \cdot \rangle$ , i.e.,

$$H^\perp = \{x \in \mathfrak{g} \mid \langle x, y \rangle = 0 \text{ for any } y \in \mathfrak{h}\}.$$

Let  $Ann_R(\mathfrak{g})$  denote

$$Ann_R(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \nabla_Y X = 0 \text{ for any } Y \in \mathfrak{g}\}$$

and let  $Ann(\mathfrak{g})$  denote

$$Ann(\mathfrak{g}) = \{X \in \mathfrak{g} \mid \nabla_X Y = \nabla_Y X = 0 \text{ for any } Y \in \mathfrak{g}\}.$$

**Remark 1.1.** Let  $M$  be a pseudo-Riemannian manifold and let  $\langle \cdot, \cdot \rangle$  be the corresponding pseudo-Riemannian metric. A vector field  $X$  on  $M$  is called a **Killing vector field** if

$$\langle \nabla_V X, W \rangle + \langle V, \nabla_W X \rangle = 0$$

holds for any vector fields  $W$  and  $V$  on  $M$ . In particular, we restrict the manifold to be a group manifold and discuss the Levi-Civita connection. Then for any vector field  $W$ ,  $W = \sum_{i=1}^n f_i X_i$ , where  $f_i \in C^\infty(G)$ ,  $n = \dim \mathfrak{g}$  and every  $X_i$  is a left invariant vector field on  $G$ . It is clear that every vector field in  $Ann_R(\mathfrak{g})$  and  $Ann(\mathfrak{g})$  is a left invariant Killing vector field.

**Proposition 1.2.**  $Ann_R(\mathfrak{g}) = (\nabla_{\mathfrak{g}}\mathfrak{g})^\perp$ .

*Proof.* Assume that  $X \in Ann_R(\mathfrak{g})$ . That is,  $\nabla_Y X = 0$  for any  $Y \in \mathfrak{g}$ . Then  $\langle \nabla_Y X, Z \rangle = 0$  for any  $Y, Z \in \mathfrak{g}$ . It follows that  $\langle X, \nabla_Y Z \rangle = 0$  for any  $Y, Z \in \mathfrak{g}$ . That is,  $Ann_R(\mathfrak{g}) \subseteq (\nabla_{\mathfrak{g}}\mathfrak{g})^\perp$ . Similarly,  $(\nabla_{\mathfrak{g}}\mathfrak{g})^\perp \subseteq Ann_R(\mathfrak{g})$ . Then  $Ann_R(\mathfrak{g}) = (\nabla_{\mathfrak{g}}\mathfrak{g})^\perp$ .  $\square$

**Proposition 1.3.** Let  $\mathfrak{h}$  be a strong ideal of  $\mathfrak{g}$ . Then  $\nabla_{\mathfrak{g}}\mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp$  and  $\nabla_{\mathfrak{h}}\mathfrak{h}^\perp = 0$ .

*Proof.* For any  $X \in \mathfrak{g}$ ,  $Y \in \mathfrak{h}$  and  $Z \in \mathfrak{h}^\perp$ ,

$$\langle Y, \nabla_X Z \rangle = -\langle \nabla_X Y, Z \rangle = 0.$$

It follows that  $\nabla_{\mathfrak{g}}\mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp$ . Also by

$$\langle X, \nabla_Y Z \rangle = -\langle \nabla_Y X, Z \rangle = 0,$$

we have  $\nabla_Y Z = 0$ . That is,  $\nabla_{\mathfrak{h}}\mathfrak{h}^\perp = 0$ .  $\square$

## 2. THE DECOMPOSITION AND GEOMETRY

Let  $G$  be a Lie group, let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and let  $\langle, \rangle$  be a left invariant pseudo-Riemannian metric on  $G$ .

**Definition 2.1.** If there exist non-trivial, non-degenerate and strong ideals  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  of  $\mathfrak{g}$  such that  $\mathfrak{g}$  is the direct sum of  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$ , then  $\mathfrak{g}$  is called **decomposable**, otherwise **indecomposable**. The decomposition is called **orthogonal** if  $\langle X, Y \rangle = 0$  for any  $X \in \mathfrak{h}_1$  and  $Y \in \mathfrak{h}_2$ . If  $\mathfrak{g}$  is decomposable (or indecomposable), then  $G$  is called decomposable (or indecomposable).

**Remark 2.2.** By Proposition 1.3, if  $\mathfrak{h}$  is a non-trivial, non-degenerate and strong ideal of  $\mathfrak{g}$ , then we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp \text{ and } \nabla_{\mathfrak{g}} \mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp.$$

But we can't prove that  $\mathfrak{h}^\perp$  is a strong ideal of  $\mathfrak{g}$ . That is, the definition of “decomposable” isn't equivalent with the existence of a non-trivial, non-degenerate and strong ideal.

By the definition and induction, we have the local decomposition of  $G$ .

**Proposition 2.3.** *There exist strong ideals  $\mathfrak{g}_i : 1 \leq i \leq l$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_l$ , where the restriction of  $\langle, \rangle$  on  $\mathfrak{g}_i$  is non-degenerate and every  $\mathfrak{g}_i$  is indecomposable.*

**Proposition 2.4** ([19]). *Let  $G$  be a connected Lie group, let  $\mathfrak{g}$  be the Lie algebra of  $G$ , let  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  be a direct sum of Lie ideals, and let  $H_i$  be the analytic subgroup of  $G$  whose Lie algebra is  $\mathfrak{h}_i$ . If  $G$  is simply connected, then  $G = H_1 \times H_2$ .*

By Propositions 2.3 and 2.4, we have the global decomposition of  $G$ .

**Proposition 2.5.** *If  $G$  is simply connected and connected, then there exist simply connected, normal and analytic subgroups  $G_i : 1 \leq i \leq l$  such that  $G = G_1 \times G_2 \cdots \times G_l$ , where the Lie algebra  $\mathfrak{g}_i$  of  $G_i$  is a strong ideal of  $\mathfrak{g}$  and every  $G_i$  is **totally geodesic** in  $G$ , i.e. every geodesic in  $G_i$  is a geodesic in  $G$ .*

*Proof.* We only need to show that  $G_i$  is totally geodesic. Let  $\langle, \rangle_i$  is the restriction of  $\langle, \rangle$  on  $G_i$  and let  $\nabla^i$  be the connection corresponding to the restrict metric. Since  $\mathfrak{g}_i$  is a strong ideal,  $\nabla_X Y \in \mathfrak{g}_i$  for any  $X, Y \in \mathfrak{g}_i$ . Then  $\nabla_X^i Y = \nabla_X Y$ , which implies that  $G_i$  is totally geodesic.  $\square$

More generally,

**Proposition 2.6.** *There exists a decomposition  $\tilde{G} = \tilde{G}_1 \times \tilde{G}_2 \cdots \times \tilde{G}_l$  of  $\tilde{G}$  into normal and analytic subgroups, where  $\tilde{G}$  is some cover group of  $G$ , the Lie algebra  $\mathfrak{g}_i$  of  $\tilde{G}_i$  is a strong ideal of  $\mathfrak{g}$ , and every  $\tilde{G}_i$  is totally geodesic in  $\tilde{G}$ .*

**Remark 2.7.** By the definition of the term “indecomposable”, there can exist a Lie group  $G$  with a left invariant Riemannian metric  $\langle, \rangle$  satisfying the following conditions:

- (1)  $G$  is indecomposable;
- (2) There exist Lie ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , where  $\mathfrak{g}$  denotes the Lie algebra of  $G$ .

Let  $G_i$  be the normal and analytic subgroup of  $G_i$  with the Lie algebra  $\mathfrak{g}_i$ . That is,  $G_i$  is a sub-manifold and not totally geodesic. Let  $\nabla^i$  be the Levi-Civita connection corresponding to the restriction of  $\langle, \rangle$  on  $G_i$ . For any  $e_i \in \mathfrak{g}_i, y \in \mathfrak{g}$ ,

$$\langle \nabla_{e_i}^i e_i - \nabla_{e_i} e_i, y \rangle = \langle \nabla_{e_i}^i e_i - \nabla_{e_i} e_i, y_1 + y_2 \rangle = \langle \nabla_{e_i} e_i, y_2 \rangle = 0,$$

which implies that  $G_i$  is **minimal**. Furthermore if  $\langle [\mathfrak{g}_1, \mathfrak{g}_1], \mathfrak{g}_2 \rangle = 0$ , it is easy to check

$$\langle \nabla_X^1 Y - \nabla_X Y, Z \rangle = 0, \text{ for any } Z \in \mathfrak{g},$$

which implies that  $G_1$  is totally geodesic. Thus if  $G$  is semi-simple,  $G_1$  is totally geodesic, if and only if  $\langle \mathfrak{g}_1, \mathfrak{g}_2 \rangle = 0$ , if and only if  $G_2$  is totally geodesic.

## 3. THE UNIQUENESS OF THE DECOMPOSITION

In this section, we will discuss the uniqueness of the decomposition given in Section 2. By the discussion in Section 2, it is enough to discuss the unique in the local decomposition.

**3.1. The case when  $\text{Ann}_R(\mathfrak{g})$  isn't isotropic.** If  $\text{Ann}_R(\mathfrak{g}) = \mathfrak{g}$ , i.e.  $\nabla_X Y = 0$  for any  $X, Y \in \mathfrak{g}$ , then there exists an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$$

such that  $\dim \mathfrak{g}_i = 1$ .

If  $\text{Ann}_R(\mathfrak{g}) \neq \mathfrak{g}$  and  $\text{Ann}_R(\mathfrak{g})$  isn't isotropic, then there exists a maximal subspace  $\mathfrak{h}_1$  of  $\text{Ann}_R(\mathfrak{g})$  such that  $\langle \cdot, \cdot \rangle|_{\mathfrak{h}_1 \times \mathfrak{h}_1}$  is non-degenerate. Let  $\mathfrak{g}_1 = \mathfrak{h}_1^\perp \neq 0$ . Then  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{g}_1$ , and for any  $X \in \mathfrak{g}$ ,  $Y \in \mathfrak{h}_1$ , and  $Z \in \mathfrak{g}_1$ ,

$$\langle Y, \nabla_X Z \rangle = -\langle \nabla_X Y, Z \rangle = 0.$$

It follows that  $\nabla_{\mathfrak{g}} \mathfrak{g}_1 \subseteq \mathfrak{g}_1$ . By  $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{g}_1$ , we have  $\nabla_{\mathfrak{g}_1} \mathfrak{g} \subseteq \mathfrak{g}_1$ . That is,  $\mathfrak{g}_1$  is a strong ideal of  $\mathfrak{g}$ . By induction, there exists a sequence of non-degenerate subalgebras of  $\mathfrak{g}$  such that

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n,$$

where  $\mathfrak{g}_i$  is a strong ideal of  $\mathfrak{g}_{i-1}$ , the quotient algebra  $\mathfrak{g}_{i-1}/\mathfrak{g}_i$  is abelian for each  $i \in \{1, 2, \dots, n\}$ , and  $\text{Ann}_R(\mathfrak{g}_n)$  is isotropic. Therefore,

**Theorem 3.1.** *Assume that  $\text{Ann}_R(\mathfrak{g}) \neq \mathfrak{g}$  and  $\text{Ann}_R(\mathfrak{g})$  isn't isotropic. Then there exists a sequence of analytic subgroups  $G_i$  of  $G$  satisfying the following conditions:*

- (1)  $G = G_0 \supset G_1 \supset \cdots \supset G_n$ ;
- (2) every  $G_i$  is a normal subgroup of  $G_{i-1}$ , and totally geodesic in  $G_{i-1}$ ;
- (3) the quotient group  $G_{i-1}/G_i$  is abelian, and  $\text{Ann}_R(\mathfrak{g}_n)$  is isotropic;
- (4) the restriction of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}_i$  is non-degenerate and every  $\mathfrak{g}_i$  is a strong ideal of  $\mathfrak{g}_{i-1}$ .

Here  $\mathfrak{g}_i$  is the Lie algebra of  $G_i$ .

**3.2. The case when  $\text{Ann}_R(\mathfrak{g})$  is isotropic.** Assume that  $\text{Ann}_R(\mathfrak{g})$  is isotropic. Let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_m$$

be decompositions of  $\mathfrak{g}$  into indecomposable, non-degenerate and strong ideals.

Firstly  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 \neq 0$ . In fact, if  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = 0$ , then  $\mathfrak{g}_1 \subseteq \text{Ann}_R(\mathfrak{g})$ , which contradicts that  $\text{Ann}_R(\mathfrak{g})$  is isotropic. Since  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \bigoplus_{j=1}^m \nabla_{\mathfrak{g}_1} \mathfrak{g}'_j$ , we have  $\nabla_{\mathfrak{g}_1} \mathfrak{g}'_j \neq 0$  for some  $j$ . Without loss of generality, assume that  $\nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 \neq 0$ .

Let  $\mathfrak{h}_1 = \bigoplus_{j=2}^n \mathfrak{g}_j$  and  $\mathfrak{h}'_1 = \bigoplus_{j=2}^m \mathfrak{g}'_j$ . We can show that the restrictions of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{h}_1$  and  $\mathfrak{h}'_1$  are non-degenerate. In a general way, we have the following proposition.

**Proposition 3.2.** *If there exist strong ideals  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ , then the restrictions of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{m}_1$  and  $\mathfrak{m}_2$  are non-degenerate.*

*Proof.* If not, assume that the restriction of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{m}_1$  is degenerate. Then there exists a non-zero vector field  $X \in \mathfrak{m}_1$  such that  $\langle X, Y \rangle = 0$  for any  $Y \in \mathfrak{m}_1$ . If  $X \in \nabla_{\mathfrak{m}_1} \mathfrak{m}_1$ , then

$$\langle X, Z \rangle = \sum \langle \nabla_{X_i} X'_i, Z \rangle = - \sum \langle X'_i, \nabla_{X_i} Z \rangle = 0$$

for any  $Z \in \mathfrak{m}_2$ . Then  $\langle X, Y \rangle = 0$  for any  $Y \in \mathfrak{g}$ . Thus  $X = 0$  since  $\langle \cdot, \cdot \rangle$  is non-degenerate. It is a contradiction, so  $X \notin \nabla_{\mathfrak{m}_1} \mathfrak{m}_1$ . Since  $\text{Ann}_R(\mathfrak{g})$  is isotropic, we have  $\text{Ann}_R(\mathfrak{g}) \subseteq \text{Ann}_R(\mathfrak{g})^\perp = \nabla_{\mathfrak{g}} \mathfrak{g}$  by Proposition 1.2. Thus  $X \notin \text{Ann}_R(\mathfrak{g})$ . Namely, there exists  $Y \in \mathfrak{m}_1$  such that  $\nabla_Y X \neq 0$ . Therefore there exists  $Z \in \mathfrak{g}$  such that  $\langle \nabla_Y X, Z \rangle \neq 0$ . Thus we have

$$\langle X, \nabla_Y Z \rangle = -\langle \nabla_Y X, Z \rangle \neq 0.$$

It contradicts the choice of  $X$  since  $\nabla_Y Z \in \mathfrak{m}_1$ . Namely, the restriction of  $\langle, \rangle$  on  $\mathfrak{m}_1$  is non-degenerate. Similarly, the restriction of  $\langle, \rangle$  on  $\mathfrak{m}_2$  is non-degenerate.  $\square$

**Lemma 3.3.**  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$  and  $\mathfrak{g}'_1 \cap \mathfrak{h}_1 = 0$ .

*Proof.* Let  $\mathfrak{b}_1 = \mathfrak{g}_1 \cap \mathfrak{g}'_1$  and  $\mathfrak{b}_2 = \mathfrak{g}_1 \cap \mathfrak{h}'_1$ . Clearly,

$$\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}_1} \mathfrak{g} = \nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 \oplus \nabla_{\mathfrak{g}_1} \mathfrak{h}'_1 \subseteq \mathfrak{b}_1 \oplus \mathfrak{b}_2.$$

(1) If  $\mathfrak{g}_1 = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ , then both  $\mathfrak{b}_1$  and  $\mathfrak{b}_2$  are non-degenerate and strong ideals of  $\mathfrak{g}_1$  by Proposition 3.2. Since  $\mathfrak{g}_1$  is indecomposable and  $\mathfrak{b}_1 \neq 0$ , we have  $\mathfrak{b}_2 = 0$ . That is,  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ .

(2) If  $\mathfrak{g}_1 \neq \mathfrak{b}_1 \oplus \mathfrak{b}_2$ , there exists  $X \in \mathfrak{g}_1$  such that  $X \notin \mathfrak{b}_1 \oplus \mathfrak{b}_2$ . Then  $X = X_1 + X_2$ , where  $X_1 \in \mathfrak{g}'_1, X_2 \in \mathfrak{h}'_1$ . Using the other decomposition,  $X_1 = X_1^1 + X_1^2$  and  $X_2 = X_2^1 + X_2^2$ , where  $X_1^1, X_2^1 \in \mathfrak{g}_1, X_1^2, X_2^2 \in \mathfrak{h}_1$ . Then  $X = X_1^1 + X_1^2 + X_2^1 + X_2^2$ . It follows that

$$X = X_1^1 + X_2^1 \text{ and } X_1^2 + X_2^2 = 0.$$

One can easily check that

$$\nabla_{\mathfrak{g}_1} X_1^1 \subseteq \nabla_{\mathfrak{g}_1} \mathfrak{g}'_1, \quad \nabla_{X_1^1} \mathfrak{g}_1 \subseteq \nabla_{\mathfrak{g}'_1} \mathfrak{g}_1; \quad \nabla_{\mathfrak{g}_1} X_2^1 \subseteq \nabla_{\mathfrak{g}_1} \mathfrak{h}'_1, \quad \nabla_{X_2^1} \mathfrak{g}_1 \subseteq \nabla_{\mathfrak{h}'_1} \mathfrak{g}_1.$$

If  $X_1^1 \notin \mathfrak{b}_1 \oplus \mathfrak{b}_2$ , let  $\mathfrak{b}_1^{(1)} = \mathfrak{b}_1 + \mathbb{R}X_1^1$  and  $\mathfrak{b}_2^{(1)} = \mathfrak{b}_2$ . If  $X_1^1 \in \mathfrak{b}_1 \oplus \mathfrak{b}_2$ , then  $X_2^1 \notin \mathfrak{b}_1 \oplus \mathfrak{b}_2$ . Let  $\mathfrak{b}_1^{(1)} = \mathfrak{b}_1$  and  $\mathfrak{b}_2^{(1)} = \mathfrak{b}_2 + \mathbb{R}X_2^1$ . It is clear that both  $\mathfrak{b}_1^{(1)}$  and  $\mathfrak{b}_2^{(1)}$  are strong ideals of  $\mathfrak{g}_1$  and  $\mathfrak{b}_1^{(1)} \cap \mathfrak{b}_2^{(1)} = 0$ . If  $\mathfrak{g}_1 = \mathfrak{b}_1^{(1)} \oplus \mathfrak{b}_2^{(1)}$ , then  $\mathfrak{b}_2^{(1)} = 0$  by Proposition 3.2. In particular,  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ . If  $\mathfrak{g}_1 \neq \mathfrak{b}_1^{(1)} \oplus \mathfrak{b}_2^{(1)}$ , since  $\dim A_1 < \infty$ , repeating the discussion in (2), we may choose  $\mathfrak{b}_1^{(k)}$  and  $\mathfrak{b}_2^{(k)}$  such that  $\mathfrak{g}_1 = \mathfrak{b}_1^{(k)} \oplus \mathfrak{b}_2^{(k)}$ , where both  $\mathfrak{b}_1^{(k)}$  and  $\mathfrak{b}_2^{(k)}$  are strong ideals of  $\mathfrak{g}_1$ . By Proposition 3.2,  $\mathfrak{b}_2^{(k)} = 0$ . In particular,  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ .

In any case we have  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ . Similarly,  $\mathfrak{g}'_1 \cap \mathfrak{h}_1 = 0$ .  $\square$

**Lemma 3.4.** The projection  $\pi_1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}'_1$  is a strong isometry from  $\mathfrak{g}$  to  $\mathfrak{g}_1$ .

*Proof.* Since  $\ker \pi_1 \subseteq \mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ , we have that  $\pi_1$  is injective. Thus  $\dim \mathfrak{g}_1 \leq \dim \mathfrak{g}'_1$ . Similarly,  $\dim \mathfrak{g}'_1 \leq \dim \mathfrak{g}_1$ . Therefore  $\dim \mathfrak{g}'_1 = \dim \mathfrak{g}_1$ . For any  $X, Y \in \mathfrak{g}_1$ , it is clear that  $\pi_1(\nabla_X Y) = \nabla_{\pi_1(X)} \pi_1(Y)$ , i.e.  $\pi_1$  is a strong isomorphism from  $\mathfrak{g}_1$  to  $\mathfrak{g}'_1$ . For any  $X \in \mathfrak{g}_1$ ,  $X = X_1 + X_2$ , where  $X_1 \in \mathfrak{g}'_1, X_2 \in \mathfrak{h}'_1$ . It is clear that  $\nabla_{\mathfrak{g}'_1} X_2 = 0$  and

$$\nabla_{\mathfrak{h}'_1} X_2 = \nabla_{\mathfrak{h}'_1} X \subseteq \mathfrak{h}'_1 \cap \mathfrak{g}_1 = 0.$$

Thus  $X_2 \in \text{Ann}_R(\mathfrak{g})$ . Therefore  $\langle X_2, X_2 \rangle = 0$ . Let  $X_1 = Y_1 + Y_2$ , where  $Y_1 \in \mathfrak{h}'_1, Y_2 \in (\mathfrak{h}'_1)^\perp$ . Furthermore  $Y_1 \in \text{Ann}_R(\mathfrak{h}'_1) \subseteq \text{Ann}_R(\mathfrak{g})$  by  $\nabla_{\mathfrak{h}'_1} Y_1 = \nabla_{\mathfrak{h}'_1} (X_1 - Y_2) = 0$ . Then

$$\langle X_1, X_2 \rangle = \langle Y_1 + Y_2, X_2 \rangle = \langle Y_2, X_2 \rangle = 0.$$

Thus  $\langle X, X \rangle = \langle X_1, X_1 \rangle = \langle \pi_1(X), \pi_1(X) \rangle$ . That is,  $\pi_1$  is a strong isometry from  $\mathfrak{g}$  to  $\mathfrak{g}_1$ .  $\square$

Furthermore, we have

$$\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{g}'_1 \text{ and } \nabla_{\mathfrak{g}_1} \mathfrak{h}'_1 = \nabla_{\mathfrak{h}'_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{h}_1 = \nabla_{\mathfrak{h}_1} \mathfrak{g}'_1 = 0.$$

Repeating the above discussion for  $j = 2, 3, \dots, n$ , we have

**Theorem 3.5.** Assume that  $\text{Ann}_R(\mathfrak{g})$  is isotropic and let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n = \mathfrak{g}'_1 \oplus \dots \oplus \mathfrak{g}'_m$$

be decompositions of  $\mathfrak{g}$ . Here  $\mathfrak{g}_i, \mathfrak{g}'_j, 1 \leq i \leq n, 1 \leq j \leq m$ , are indecomposable, non-degenerate and strong ideals of  $\mathfrak{g}$ . Then  $n = m$  and

(1) Changing the subscripts if necessary, we have  $\dim \mathfrak{g}_j = \dim \mathfrak{g}'_j$  and

$$\nabla_{\mathfrak{g}_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}_j} \mathfrak{g}'_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}'_j; \quad \nabla_{\mathfrak{g}_j} \mathfrak{g}'_k = \nabla_{\mathfrak{g}'_j} \mathfrak{g}_k = 0 \text{ if } j \neq k.$$

- (2) The projections  $\pi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}'_i, 1 \leq i \leq n$  are strong isometries from  $\mathfrak{g}_i$  to  $\mathfrak{g}'_i$ , so  $\pi = (\pi_1, \dots, \pi_n)$  is a strong automorphism of  $\mathfrak{g}$ .

**3.3. The case when  $\text{Ann}_R(\mathfrak{g}) = \text{Ann}(\mathfrak{g})$ .** Let  $\mathfrak{g}$  be decomposable, i.e.  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ , where  $\mathfrak{g}_i, i = 1, 2$  are non-degenerate and strong ideals of  $\mathfrak{g}$ . Therefore  $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_1^\perp$ ,  $\nabla_{\mathfrak{g}} \mathfrak{g}_1^\perp \subseteq \mathfrak{g}_1^\perp$ , and  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1^\perp = 0$  by Proposition 1.3. For any  $X \in \mathfrak{g}_1^\perp$ ,

$$X = X_1 + X_2,$$

where  $X_1 \in \mathfrak{g}_1$  and  $X_2 \in \mathfrak{g}_2$ . Since both  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are ideals, we have

$$\langle \nabla_Y X_1, Z \rangle = -\langle X_1, \nabla_Y Z \rangle = \langle X_2, \nabla_Y Z \rangle = -\langle \nabla_Y X_2, Z \rangle = 0$$

for any  $Y, Z \in \mathfrak{g}_1$ . Thus  $\nabla_{\mathfrak{g}_1} X_1 = 0$  since  $\mathfrak{g}_1$  is non-degenerate. Namely  $X_1 \in \text{Ann}_R(\mathfrak{g})$ . By the assumption,  $X_1 \in \text{Ann}(\mathfrak{g})$ . Then  $\nabla_X Y = \nabla_{(X_1+X_2)} Y = 0$  for any  $Y \in \mathfrak{g}_1$ . That is,

$$\nabla_{\mathfrak{g}_1^\perp} \mathfrak{g}_1 = 0.$$

It follows that  $\mathfrak{g}_1^\perp$  is a non-degenerate and strong ideal of  $\mathfrak{g}$ . Similarly,  $\mathfrak{g}_2^\perp$  is a non-degenerate and strong ideal of  $\mathfrak{g}$ . That is,  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp$  and  $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_2^\perp$  are orthogonal decompositions of  $\mathfrak{g}$  into non-degenerate and strong ideals.

Let  $\mathfrak{h}$  a maximal non-degenerate subspace of  $\text{Ann}_R(\mathfrak{g})$ . Then  $\mathfrak{h}$  is a strong ideal of  $\mathfrak{g}$  and  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$  as vector spaces. For any  $X \in \mathfrak{g}, Y \in \mathfrak{h}$  and  $Z \in \mathfrak{h}^\perp$ ,

$$\langle \nabla_X Z, Y \rangle = -\langle Z, \nabla_X Y \rangle = 0.$$

That is,  $\nabla_{\mathfrak{g}} \mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp$ . By  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$  and  $\text{Ann}_R(\mathfrak{g}) = \text{Ann}(\mathfrak{g})$ , we have  $\nabla_{\mathfrak{h}^\perp} \mathfrak{g} \subseteq \mathfrak{h}^\perp$ . Namely  $\mathfrak{h}^\perp$  is a non-degenerate and strong ideal of  $\mathfrak{g}$ . It is easy to see that  $\text{Ann}_R(\mathfrak{h}^\perp)$  is isotropic.

**Proposition 3.6.** Assume that  $\text{Ann}_R(\mathfrak{g}) = \text{Ann}(\mathfrak{g})$ . There exists an orthogonal decomposition  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_l$  of  $\mathfrak{g}$  satisfying the following conditions.

- (1)  $\mathfrak{g}_0$  is a maximal non-degenerate subspace of  $\text{Ann}_R(\mathfrak{g})$ ,
- (2)  $\mathfrak{g}_i$  is an indecomposable, non-degenerate and strong ideal of  $\mathfrak{g}$  for any  $1 \leq i \leq l$ ,
- (3)  $\text{Ann}_R(\mathfrak{g}_i)$  is isotropic for any  $1 \leq i \leq l$ .

**Remark 3.7.** If  $\text{Ann}_R(\mathfrak{g}) \neq \text{Ann}(\mathfrak{g})$ , then there exists  $\mathfrak{g}$  which is decomposable and any decomposition of which isn't orthogonal. In fact, let  $G$  be a Lie group, let  $\mathfrak{g}$  be the Lie algebra of  $G$ , let  $\langle, \rangle$  be a left invariant pseudo-Riemannian metric on  $G$ , and there exists a basis  $\{X_1, \dots, X_4, Y_1, \dots, Y_4\}$  of  $\mathfrak{g}$  such that

- (1)  $\langle X_1, X_4 \rangle = \langle X_2, X_3 \rangle = \langle Y_1, Y_4 \rangle = \langle Y_2, Y_3 \rangle = \langle X_4, Y_4 \rangle = 1$ ,
- (2)  $[X_1, X_3] = X_1, [X_1, X_4] = -X_2, [Y_1, Y_3] = Y_1, [Y_1, Y_4] = -Y_2$ .

Then by the equation (1.1), the connection satisfies

$$\nabla_{X_1} X_3 = X_1, \nabla_{X_1} X_4 = -X_2, \nabla_{Y_1} Y_3 = Y_1, \nabla_{Y_1} Y_4 = -Y_2.$$

Let  $\mathfrak{g}_1$  be the Lie subalgebra extended by  $\{X_1, \dots, X_4\}$  and let  $\mathfrak{g}_2$  be the Lie subalgebra extended by  $\{Y_1, \dots, Y_4\}$ . It is clear that both  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are non-degenerate and strong ideals of  $\mathfrak{g}$ . Assume that  $\mathfrak{h}$  is a non-degenerate and strong ideal of  $\mathfrak{g}_1$ . If  $aX_1 + bX_2 + cX_3 + dX_4 \in \mathfrak{h}$  for  $a \neq 0$ , then  $X_1, X_2 \in \mathfrak{h}$  since  $\mathfrak{h}$  is a strong ideal of  $\mathfrak{g}$ . Furthermore  $\mathfrak{h} = \mathfrak{g}$  since  $\mathfrak{h}$  is non-degenerate. It follows that  $\mathfrak{g}_1$  is indecomposable. Similarly,  $\mathfrak{g}_2$  is indecomposable. Then  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  is a decomposition of  $\mathfrak{g}$  into indecomposable, non-degenerate and strong ideals. By Theorem 3.5, any decomposition of  $\mathfrak{g}$  into indecomposable, non-degenerate and strong ideals is

$$\mathfrak{g} = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2,$$

where  $\mathfrak{g}'_i = \pi_i(\mathfrak{g}_i)$ . Here  $\pi_i$  is the projection from  $\mathfrak{g}_i$  to  $\mathfrak{g}'_i$ . Then  $\{X_1, X_2, \pi_1(X_3), \pi_1(X_4)\}$  is a basis of  $\mathfrak{g}'_1$ . By the proof of Theorem 3.5,  $\pi_1(X_3) = X_3 + Y$ , where  $Y \in \text{Ann}_R(\mathfrak{g}'_2)$ . Since  $\mathfrak{g}'_1$  is a

strong ideal, we have  $Y \in \text{Ann}(\mathfrak{g}'_2)$ . Similarly  $\pi_1(X_4) - X_4 \in \text{Ann}(\mathfrak{g}'_2)$  and  $\pi_2(Y_4) - Y_4 \in \text{Ann}(\mathfrak{g}'_1)$ . It follows that

$$\langle \pi_1(X_4), \pi_2(Y_4) \rangle = \langle X_4, Y_4 \rangle = 1.$$

That is, any decomposition isn't orthogonal.

Assume that  $\text{Ann}_R(\mathfrak{g}) = \text{Ann}(\mathfrak{g})$ . Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_m$  be orthogonal decompositions of  $\mathfrak{g}$  such that, for any  $1 \leq i \leq n$  and  $1 \leq j \leq m$ ,

- (1) both  $\mathfrak{g}_0$  and  $\mathfrak{g}'_0$  are maximal non-degenerate subspaces of  $\text{Ann}_R(\mathfrak{g})$ ,
- (2) both  $\mathfrak{g}_i$  and  $\mathfrak{g}'_j$  are indecomposable, non-degenerate and strong ideals of  $\mathfrak{g}$ ,
- (3) both  $\text{Ann}_R(\mathfrak{g}_i)$  and  $\text{Ann}_R(\mathfrak{g}'_j)$  are isotropic.

Firstly it is clear that there exists a strong isometry  $\pi_0$  from  $\mathfrak{g}_0$  to  $\mathfrak{g}'_0$ . Secondly  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 \neq 0$ . In fact, if  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = 0$ , then  $\mathfrak{g}_1 = \text{Ann}_R(\mathfrak{g}_1)$ , which contradicts that  $\text{Ann}_R(\mathfrak{g}_1)$  is isotropic. Since  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}_1} \mathfrak{g} = \bigoplus_{j=1}^m \nabla_{\mathfrak{g}_1} \mathfrak{g}'_j$ , we have  $\nabla_{\mathfrak{g}_1} \mathfrak{g}'_j \neq 0$  for some  $j$ . Without loss of generality, assume that  $\nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 \neq 0$ .

Let  $\mathfrak{h}_1 = \bigoplus_{j \neq 1} \mathfrak{g}_j$  and  $\mathfrak{h}'_1 = \bigoplus_{j \neq 1} \mathfrak{g}'_j$ . It is clear that the restrictions of  $\langle, \rangle$  on  $\mathfrak{h}_1$  and  $\mathfrak{h}'_1$  are non-degenerate. Let  $\mathfrak{b}_1 = \mathfrak{g}_1 \cap \mathfrak{g}'_1$  and  $\mathfrak{b}_2 = \mathfrak{g}_1 \cap \mathfrak{h}'_1$ . Similar to the proof of Lemma 3.3, we have

$$\mathfrak{g}_1 = \mathfrak{b}_1^{(k)} \oplus \mathfrak{b}_2^{(k)},$$

where both  $\mathfrak{b}_1^{(k)}$  and  $\mathfrak{b}_2^{(k)}$  are strong ideals of  $\mathfrak{g}_1$  satisfying  $\mathfrak{b}_1 \subseteq \mathfrak{b}_1^{(k)}$  and  $\mathfrak{b}_2 \subseteq \mathfrak{b}_2^{(k)}$ . The restrictions of  $\langle, \rangle$  on  $\mathfrak{b}_1^{(k)}$  and  $\mathfrak{b}_2^{(k)}$  are non-degenerate by Proposition 3.2. Since  $\mathfrak{g}_1$  is indecomposable and  $0 \subsetneq \nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_1^{(k)}$ , we have  $\mathfrak{b}_2^{(k)} = 0$ . Then  $\mathfrak{g}_1 \cap \mathfrak{h}'_1 = 0$ . Similarly  $\mathfrak{g}'_1 \cap \mathfrak{h}_1 = 0$ . It follows that  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}_1} \mathfrak{g}'_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{g}'_1$  and  $\nabla_{\mathfrak{g}_1} \mathfrak{h}'_1 = \nabla_{\mathfrak{h}'_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{h}_1 = \nabla_{\mathfrak{h}_1} \mathfrak{g}'_1 = 0$ .

By the proof of Lemma 3.4, the projection  $\pi_1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}'_1$  is a strong isomorphism from  $\mathfrak{g}_1$  to  $\mathfrak{g}'_1$ . In particular,  $\dim \mathfrak{g}_1 = \dim \mathfrak{g}'_1$ . Since  $\text{Ann}_R(\mathfrak{g}_1)$  is isotropic, by Proposition 1.2, we have

$$\text{Ann}_R(\mathfrak{g}_1) \subseteq \text{Ann}_R(\mathfrak{g}_1)^\perp = \nabla_{\mathfrak{g}_1} \mathfrak{g}_1 = \nabla_{\mathfrak{g}'_1} \mathfrak{g}'_1.$$

Then  $\text{Ann}_R(\mathfrak{g}_1) = \text{Ann}_R(\mathfrak{g}'_1)$ , and there exists a basis  $\{X_1, \dots, X_k, \dots, X_s, \dots, X_{s+k}\}$  of  $\mathfrak{g}_1$ , where  $\{X_1, \dots, X_k\}$  is a basis of  $\text{Ann}_R(\mathfrak{g}_1)$ ,  $\{X_1, \dots, X_s\}$  is a basis of  $\nabla_{\mathfrak{g}_1} \mathfrak{g}_1$ , such that

$$\langle X_i, X_i \rangle = \varepsilon_i \text{ for } k+1 \leq i \leq s; \quad \langle X_i, X_{s+i} \rangle = 1 \text{ for } 1 \leq i \leq k,$$

and others are zero, where  $\varepsilon_i$  denotes the sign. It is easy to see that  $\pi_1|_{\nabla_{\mathfrak{g}_1} \mathfrak{g}_1} = id$  and  $\langle \pi_1(X_i), \pi_1(X_j) \rangle = \langle X_i, X_j \rangle$  for any  $1 \leq i \leq s+k$  and  $1 \leq j \leq s$ . Assume that  $X_p = X_{p_0} + X_{p_1} + X_{p_2}$  for any  $s+1 \leq p \leq s+k$ , where  $X_{p_0} \in \mathfrak{g}'_0$ ,  $X_{p_1} \in \mathfrak{g}'_1$  and  $X_{p_2} \in \bigoplus_{i=2}^m \mathfrak{g}'_i$ . For  $s+1 \leq q \leq s+k$ , we have

$$0 = \langle X_p, X_q \rangle = \langle X_{p_1}, X_{q_1} \rangle + \langle X_{p_0}, X_{q_0} \rangle.$$

Let  $b_{pq} = \langle X_{p_0}, X_{q_0} \rangle$  if  $p \neq q$ ,  $2b_{pp} = \langle X_{p_0}, X_{p_0} \rangle$  and  $X'_{p_1} = X_{p_1} + \sum_{l=p}^{s+k} b_{pl} X_{l-s}$ . Then

$$\begin{aligned} \langle X'_{p_1}, X'_{p_1} \rangle &= \langle X_{p_1}, X_{p_1} \rangle + 2b_{pp} = 0, \quad s+1 \leq p \leq s+k; \\ \langle X'_{p_1}, X'_{q_1} \rangle &= \langle X_{p_1}, X_{q_1} \rangle + b_{pq} = 0, \quad s+1 \leq p \leq q \leq s+k. \end{aligned}$$

Define  $\pi'_1 : \mathfrak{g}_1 \rightarrow \mathfrak{g}'_1$  by

$$\pi'_1(X_j) = X_j \text{ for } 1 \leq j \leq s; \quad \pi'_1(X_j) = X'_{j_1} \text{ for } s+1 \leq j \leq s+k.$$

It is easy to check that  $\pi'_1$  is a strong isometry from  $\mathfrak{g}_1$  onto  $\mathfrak{g}'_1$ . Repeating the above discussion for  $j = 2, 3, \dots, n$ , we have



**Theorem 3.8.** Assume that  $\text{Ann}_R(\mathfrak{g}) = \text{Ann}(\mathfrak{g})$  and

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_m$$

are orthogonal decompositions of  $\mathfrak{g}$ . Here  $\mathfrak{g}_0, \mathfrak{g}'_0$  are maximal and non-degenerate subspaces of  $\text{Ann}_R(\mathfrak{g})$ , and  $\mathfrak{g}_i, \mathfrak{g}'_j, 1 \leq i \leq n, 1 \leq j \leq m$ , are indecomposable, non-degenerate and strong ideals of  $\mathfrak{g}$ . Then  $n = m$  and

(1) Changing the subscripts if necessary, we have  $\dim \mathfrak{g}_j = \dim \mathfrak{g}'_j$  and

$$\nabla_{\mathfrak{g}_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}_j} \mathfrak{g}'_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}'_j; \quad \nabla_{\mathfrak{g}_j} \mathfrak{g}'_k = \nabla_{\mathfrak{g}'_j} \mathfrak{g}_k = 0 \text{ if } j \neq k.$$

(2) The projections  $\pi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}'_i, 1 \leq i \leq n$  are strong isometries from  $\mathfrak{g}_i$  to  $\mathfrak{g}'_i$ , so  $\pi = (\pi_0, \pi_1, \dots, \pi_n)$  is a strong isometry of  $\mathfrak{g}$ . That is, the decomposition is unique up to a strong isometry.

**3.4. The case when  $\text{Ann}_R(\mathfrak{g}) = 0$ .** Clearly  $\text{Ann}(\mathfrak{g}) = \text{Ann}_R(\mathfrak{g}) = 0$ .

**Proposition 3.9.** Assume that  $\text{Ann}_R(\mathfrak{g}) = 0$ . Then the following conditions are equivalent.

- (1)  $\mathfrak{g}$  is decomposable,
- (2) there exist non-trivial and strong ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ ,
- (3) there exist non-trivial Lie ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  and  $\langle X, Y \rangle = 0$  for any  $X \in \mathfrak{g}_1$  and  $Y \in \mathfrak{g}_2$ .

*Proof.* The condition (1) is equivalent with (2) by Proposition 3.2. Assume that there exist non-trivial Lie ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  and  $\langle X, Y \rangle = 0$  for any  $X \in \mathfrak{g}_1$  and  $Y \in \mathfrak{g}_2$ . For any  $X \in \mathfrak{g}, Y \in \mathfrak{g}_1$ , and  $Z \in \mathfrak{g}_2$ ,

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle) = 0.$$

That is,  $\nabla_X Y \in \mathfrak{g}_1$ . It follows that  $\mathfrak{g}_1$  is a strong ideal of  $\mathfrak{g}$ . Similarly,  $\mathfrak{g}_2$  is a strong ideal of  $\mathfrak{g}$ . Assume that there exist non-trivial and strong ideals  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ . By Proposition 1.2,  $\mathfrak{g} = \nabla_{\mathfrak{g}} \mathfrak{g}$  since  $\text{Ann}_R(\mathfrak{g}) = 0$ . Since every  $\mathfrak{g}_i$  is a strong ideal of  $\mathfrak{g}$ , we have

$$\mathfrak{g}_1 = \nabla_{\mathfrak{g}_1} \mathfrak{g}_1.$$

Thus for any  $X \in \mathfrak{g}_1$  and  $Z \in \mathfrak{g}_2$ ,

$$\langle X, Z \rangle = \sum_i \langle \nabla_{X_i} Y_i, Z \rangle = - \sum_i \langle Y_i, \nabla_{X_i} Z \rangle = 0.$$

Namely, the decomposition is orthogonal. Then the proposition follows.  $\square$

Then by Proposition 2.3, Theorem 3.8 and Proposition 3.9, we have:

**Theorem 3.10.** Assume that  $\text{Ann}_R(\mathfrak{g}) = 0$ . There exist strong ideals  $\mathfrak{g}_i : 1 \leq i \leq l$  of  $\mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_l$ , where the restriction of  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}_i$  is non-degenerate, every  $\mathfrak{g}_i$  is indecomposable and  $\text{Ann}_R(\mathfrak{g}_i) = 0$ . Furthermore,  $\langle X, Y \rangle = 0$  for any  $X \in \mathfrak{g}_i$  and  $Y \in \mathfrak{g}_j$  if  $i \neq j$ , and the decomposition is unique up to the order of strong ideals.

*Proof.* It is enough to prove that the decomposition is unique up to the order of strong ideals. The others follows from Proposition 2.3, Theorem 3.8 and Proposition 3.9. Let  $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \cdots \oplus \mathfrak{g}'_s$  be another decomposition. By Theorem 3.8,  $s = l$ ,  $\dim \mathfrak{g}_j = \dim \mathfrak{g}'_j$  changing the subscripts if necessary, and  $\nabla_{\mathfrak{g}_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}_j} \mathfrak{g}'_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}_j = \nabla_{\mathfrak{g}'_j} \mathfrak{g}'_j$ . Since  $\text{Ann}_R(\mathfrak{g}) = 0$ , and both  $\mathfrak{g}_i$  and  $\mathfrak{g}'_j$  are strong ideals of  $\mathfrak{g}$ , we have  $\mathfrak{g}_i = \nabla_{\mathfrak{g}_i} \mathfrak{g}_i = \nabla_{\mathfrak{g}'_i} \mathfrak{g}'_i = \mathfrak{g}'_i$ . That is, the decomposition is unique up to the order of strong ideals.  $\square$

**Remark 3.11.** We can give a direct proof of Theorem 3.10 as follows. Assume that  $\mathfrak{g}$  is decomposable and  $\text{Ann}_R(\mathfrak{g}) = 0$ . By Proposition 3.9, then there exist Lie ideals  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  of  $\mathfrak{g}$  such that

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \text{ and } \langle \mathfrak{h}_1, \mathfrak{h}_2 \rangle = 0.$$

Let  $\langle, \rangle_i = \langle, \rangle|_{\mathfrak{h}_i \times \mathfrak{h}_i}$  for  $i = 1, 2$ . Then we have an orthogonal decomposition  $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$  of  $\mathfrak{g}$ , where every  $\mathfrak{g}_i$  is an indecomposable, non-degenerate and strong ideal by Proposition 3.9.

Assume that  $\mathfrak{g} = \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_m$  is another orthogonal decomposition of  $\mathfrak{g}$ . Here  $\mathfrak{g}'_j$ ,  $1 \leq j \leq m$ , are indecomposable, non-degenerate and strong ideals of  $\mathfrak{g}$ . Since  $\mathfrak{g} = \nabla_{\mathfrak{g}}\mathfrak{g}$  and every  $\mathfrak{g}_i$  is a strong ideal of  $\mathfrak{g}$ , we have

$$\mathfrak{g}_1 = \nabla_{\mathfrak{g}_1}\mathfrak{g}_1.$$

Then  $\nabla_{\mathfrak{g}_1}\mathfrak{g}'_j \neq 0$  for some  $j$  since  $\nabla_{\mathfrak{g}_1}\mathfrak{g}_1 = \nabla_{\mathfrak{g}_1}\mathfrak{g} = \bigoplus_{j=1}^m \nabla_{\mathfrak{g}_1}\mathfrak{g}'_j$ . Without loss of generality, assume that  $\nabla_{\mathfrak{g}_1}\mathfrak{g}'_1 \neq 0$ . Let  $\mathfrak{h}_1 = \bigoplus_{j=2}^n \mathfrak{g}_j$ ,  $\mathfrak{h}'_1 = \bigoplus_{j=2}^m \mathfrak{g}'_j$ ,  $\mathfrak{b}_1 = \mathfrak{g}_1 \cap \mathfrak{g}'_1$  and  $\mathfrak{b}_2 = \mathfrak{g}_1 \cap \mathfrak{h}'_1$ . Clearly,

$$\mathfrak{g}_1 = \nabla_{\mathfrak{g}_1}\mathfrak{g}_1 = \nabla_{\mathfrak{g}_1}\mathfrak{g} = \nabla_{\mathfrak{g}_1}\mathfrak{g}'_1 \oplus \nabla_{\mathfrak{g}_1}\mathfrak{h}'_1 \subseteq \mathfrak{b}_1 \oplus \mathfrak{b}_2 \subseteq \mathfrak{g}_1.$$

That is,  $\mathfrak{g}_1 = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ . Since  $\mathfrak{g}_1$  is indecomposable and  $\mathfrak{b}_1 \neq 0$ , we have  $\mathfrak{b}_2 = 0$ . It follows that  $\mathfrak{g}_1 = \mathfrak{b}_1 = \mathfrak{g}_1 \cap \mathfrak{g}'_1$ . That is,  $\mathfrak{g}_1 \subseteq \mathfrak{g}'_1$ . Similarly,  $\mathfrak{g}'_1 \subseteq \mathfrak{g}_1$ . Thus  $\mathfrak{g}_1 = \mathfrak{g}'_1$ . Then the theorem follows from repeating the above discussion for  $j = 2, 3, \dots, n$ .

#### 4. SOME APPLICATIONS OF THE DECOMPOSITION RESULTS

The **curvature tensor** associated with  $\langle, \rangle$  is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z.$$

If  $R = 0$ , then  $\langle, \rangle$  is said to be **flat**. For any pair  $X, Y \in \mathfrak{g}$ , define the **Ricci tensor**  $\text{ric}_{\langle, \rangle}$  associated with  $\langle, \rangle$  by

$$\text{ric}_{\langle, \rangle}(X, Y) = \text{Tr}\{Z \mapsto R(Z, X)Y\}.$$

The pseudo-Riemannian metric  $\langle, \rangle$  is said to be **Einstein** if

$$\text{ric}_{\langle, \rangle} = c\langle, \rangle$$

for some constant  $c$ . In particular,  $\langle, \rangle$  is said to be **Ricci flat** if  $c = 0$ .

In [3], there are some results on Lie groups with flat left invariant pseudo-Riemannian metrics. Flat pseudo-Riemannian metrics are Ricci flat, which are a class of pseudo-Riemannian Einstein metrics. On the other hand,

**Example 4.1.** Let  $G$  be a Lie group, let  $\mathfrak{g}$  be its Lie algebra, let  $\langle, \rangle$  be a bi-invariant pseudo-Riemannian metric on  $G$ , and let  $K$  be the Killing form of  $\mathfrak{g}$ . Then

$$\nabla_X Y = \frac{1}{2}[X, Y] \text{ for any } X, Y \in \mathfrak{g}$$

and the curvature tensor

$$R(X, Y) = -\frac{1}{4}\text{ad}[X, Y].$$

Clearly,  $\langle, \rangle$  is flat if and only if  $\mathfrak{g}$  is two-step nilpotent (also see [3]). Let  $\{E_1, \dots, E_n\}$  be a basis of  $\mathfrak{g}$  and let  $\{E_1^*, \dots, E_n^*\}$  be the dual basis corresponding to  $\langle, \rangle$ . Then for any  $X, Y \in \mathfrak{g}$ ,

$$\begin{aligned} \text{ric}_{\langle, \rangle}(X, Y) &= \sum_{i=1}^n \langle R(E_i, X)Y, E_i^* \rangle = -\frac{1}{4} \sum_{i=1}^n \langle \text{ad}[E_i, X](Y), E_i^* \rangle \\ &= -\frac{1}{4} \sum_{i=1}^n \langle \text{ad}Y \text{ad}X(E_i), E_i^* \rangle = -\frac{1}{4} \text{Tr}(\text{ad}Y \text{ad}X) \\ &= -\frac{1}{4} K(X, Y). \end{aligned}$$

In addition, assume that  $\langle, \rangle$  is an Einstein metric with the constant  $c$ . That is,

$$-\frac{1}{4}K(X, Y) = c\langle X, Y \rangle.$$

- (1) If  $c \neq 0$ , then  $K$  is non-degenerate. It follows that  $G$  is semisimple.
- (2) If  $c = 0$ , i.e.  $\langle, \rangle$  is Ricci flat, then  $G$  is solvable.

In particular, if  $\mathfrak{g}$  is at least 3-step nilpotent, then  $\langle, \rangle$  is not flat but Ricci flat. For this case, we have  $\text{Ann}_R(\mathfrak{g}) = C(\mathfrak{g}) \neq 0$ .

**Proposition 4.2.** *If  $\text{ric}_{\langle, \rangle}$  is non-degenerate, then  $\text{Ann}_R(\mathfrak{g}) = 0$  and  $\mathfrak{g} = \nabla_{\mathfrak{g}}\mathfrak{g}$ .*

*Proof.* Assume that  $X \in \text{Ann}_R(\mathfrak{g})$ . Then for any  $Y \in \mathfrak{g}$ ,

$$\begin{aligned} \text{ric}_{\langle, \rangle}(Y, X) &= \sum_{i=1}^m \langle R(E_i, Y)X, E_i^* \rangle = \sum_{i=1}^m \langle \nabla_{E_i} \nabla_Y X - \nabla_Y \nabla_{E_i} X - \nabla_{[E_i, Y]} X, E_i^* \rangle \\ &= 0, \end{aligned}$$

where  $E_i$  is any basis of  $\mathfrak{g}$ ,  $E_i^*$  is the dual basis of  $E_i$  corresponding to  $\langle, \rangle$ , and  $R$  is the curvature tensor. Since  $\text{ric}_{\langle, \rangle}$  is non-degenerate, we have  $X = 0$ . That is,

$$\text{Ann}_R(\mathfrak{g}) = 0.$$

By Proposition 1.2,  $\mathfrak{g} = \nabla_{\mathfrak{g}}\mathfrak{g}$ . □

Then by Theorem 3.10,

**Theorem 4.3.** *Let  $G$  be a Lie group, let  $\mathfrak{g}$  be its Lie algebra, and let  $\langle, \rangle$  be a left invariant pseudo-Riemannian metric on  $G$  such that  $\text{ric}_{\langle, \rangle}$  is non-degenerate. Then there exist strong ideals  $\mathfrak{g}_i : 1 \leq i \leq l$  of  $\mathfrak{g}$  such that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l,$$

where the restriction of  $\langle, \rangle$  on  $\mathfrak{g}_i$  is non-degenerate and  $\mathfrak{g}_i$  is indecomposable for any  $1 \leq i \leq l$ . Furthermore  $\langle X, Y \rangle = 0$  and  $\text{ric}_{\langle, \rangle}(X, Y) = 0$  for any  $X \in \mathfrak{g}_i$  and  $Y \in \mathfrak{g}_j$  if  $i \neq j$ , and the decomposition is unique up to the order of the ideals.

*Proof.* For any  $X \in \mathfrak{g}_i$  and  $Y \in \mathfrak{g}_j$  if  $i \neq j$ ,

$$\begin{aligned} \text{ric}_{\langle, \rangle}(X, Y) &= \sum_{i=1}^m \langle R(E_i, X)Y, E_i^* \rangle = \sum_{i=1}^m \langle \nabla_{E_i} \nabla_X Y - \nabla_X \nabla_{E_i} Y - \nabla_{[E_i, X]} Y, E_i^* \rangle \\ &= 0 \end{aligned}$$

since both  $\mathfrak{g}_i$  and  $\mathfrak{g}_j$  are strong ideals. The others follow from Theorem 3.10. □

In particular, assume that  $\langle, \rangle$  is a left invariant pseudo-Riemannian Einstein metric with a non-zero constant  $c$ . Clearly  $\text{ric}_{\langle, \rangle}$  is non-degenerate. By Theorem 4.3, we have:

**Corollary 4.4.** *Let  $G$  be a Lie group, let  $\mathfrak{g}$  be its Lie algebra, and let  $\langle, \rangle$  be a left invariant pseudo-Riemannian Einstein metric on  $G$  with a non-zero constant  $c$ . Then there exist strong ideals  $\mathfrak{g}_i : 1 \leq i \leq l$  of  $\mathfrak{g}$  such that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l,$$

where the restriction of  $\langle, \rangle$  on  $\mathfrak{g}_i$  is non-degenerate and  $\mathfrak{g}_i$  is indecomposable for any  $1 \leq i \leq l$ . Furthermore  $\langle X, Y \rangle = 0$  for any  $X \in \mathfrak{g}_i$  and  $Y \in \mathfrak{g}_j$  if  $i \neq j$ , the restriction of  $\langle, \rangle$  on  $\mathfrak{g}_i$  is a left invariant pseudo-Riemannian Einstein metric with the same constant  $c$ , and the decomposition is unique up to the order of the ideals.

It is shown in Example 4.1 that Ricci flat doesn't imply flat for a left invariant pseudo-Riemannian metric. But for a Riemannian metric, it is proved in [1] that  $\langle, \rangle$  is Ricci flat if and only if  $\langle, \rangle$  is flat. At the end, we give the well-known structure of a Lie group with a left invariant flat Riemannian metric.

**Theorem 4.5** ([24], Theorem 1.5). *A Lie group  $G$  with a left invariant Riemannian metric is flat if and only if its Lie algebra  $\mathfrak{g}$  splits as an orthogonal direct sum  $\mathfrak{b} \oplus \mathfrak{u}$ , where  $\mathfrak{b}$  is a commutative subalgebra,  $\mathfrak{u}$  is a commutative ideal, and where the linear transformation  $\text{ad } b$  is skew-adjoint for any  $b \in \mathfrak{b}$ . Furthermore, if these conditions are satisfied, then*

$$\nabla_u = 0, \nabla_b = \text{ad}(b), \text{ for any } u \in \mathfrak{u}, b \in \mathfrak{b}. \quad (4.1)$$

In order to describe  $\text{Ann}_R(\mathfrak{g})$  clearly, we can rewrite Theorem 3.9 in [10].

**Theorem 4.6.** *A Lie group  $G$  with a left invariant Riemannian metric is flat if and only if its Lie algebra  $\mathfrak{g}$  splits as an orthogonal direct sum  $\mathfrak{b} \oplus \text{Ann}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ , where  $\mathfrak{b}$  is a commutative subalgebra,  $[\mathfrak{g}, \mathfrak{g}] = \nabla_{\mathfrak{g}} \mathfrak{g}$  is a commutative ideal with even dimension,  $\dim \mathfrak{b} \leq \dim[\mathfrak{g}, \mathfrak{g}]/2$ , and  $\nabla b = 0$  and  $\nabla_b = \text{ad}(b)$  is skew-adjoint for any non-zero  $b \in \mathfrak{b}$ .*

**Remark 4.7.** By Theorem 4.6,  $\text{Ann}_R(\mathfrak{g}) = \mathfrak{b} \oplus \text{Ann}(G) \neq 0$ . Clearly, the decomposition isn't unique if  $\dim \text{Ann}(G) \geq 2$ . That is, the uniqueness in Corollary 4.4 doesn't hold for flat metrics.

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